Characterization of eroders in partially ordered one-dimensional cellular automata

Mathieu Hilaire
École normale supérieure Paris-Saclay
Boston University, Boston
`Correspondence: mathieu.hilaire@ens-paris-saclay.fr`

Introduction

In computability theory, it is important to study cellular automata and how they work. One of the aspects of this field of study is the study of automata with certain probability of error, as it reflect the real life case were components of a computational system might make a mistake.

When studying this kind of automata, the issue of ergodicity arose: in an ergodic process, the state of the process after a long time is nearly independent of its initial state. This is an issue as we would like the computations of the automaton to be different depending on the entries of the automaton.

It is therefore important to study infinite cellular automata and conditions under which they are non-ergodic. In this internship, we decided to study some property that guarantees that cellular automata are "deterministic enough" to stay non-ergodic even in cases of probabilistic error occurring.

This is where the notions of stable trajectories and attractor is important. Here, we shall study deterministic cases only, as there still are plenty of unsolved problems in this area. Our main objective is to improve the characterizations of eroders (when 'all zeroes' is an attractor), and we study initially the case of the plane and then systems with multiples lines.

State of the Art

For any one dimensional monotonic (with space totally ordered) standard deterministic system, we know a characterization of all the eroders, as seen in Gal'perin's Paper (See [Gal76]). For any two dimensional monotonic standard deterministic system with only 0 and 1 as states, there also exists a characterization (See [Too80] and [Too82]). This characterization can be extended to any d-dimensional commutative monotonic standard deterministic system with set of states 0 and 1 (See [Too86]). Because of this, we know what are the conditions for islands of 1 to be eroded by seas of 0 in theses systems.

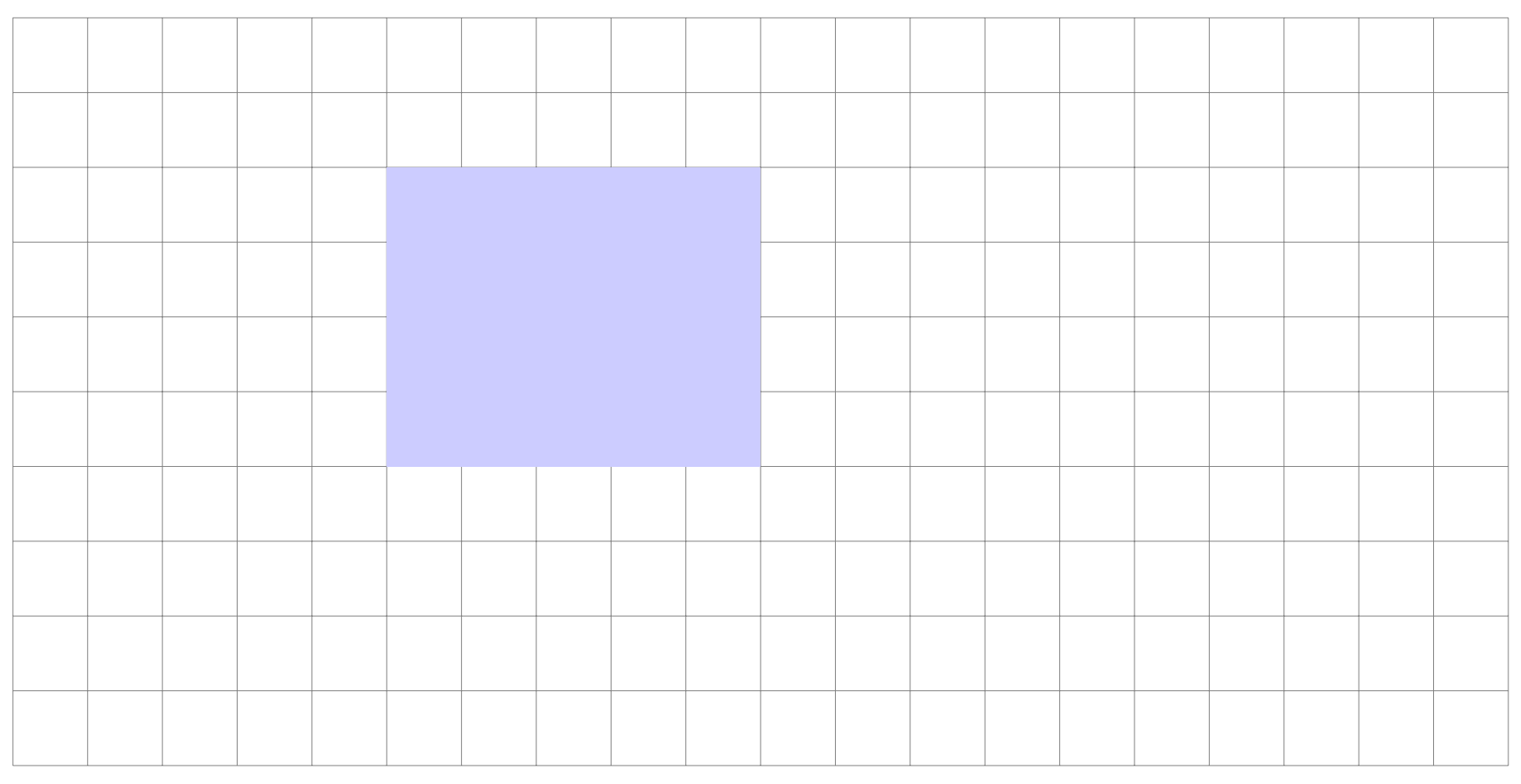

Figure 1: an island of 1 (in blue) in a sea of 0 (in white) in a two dimensional cellular automaton

Our initial aim being to characterize eroders in the more general two dimensional case, we had to consider cases where there are more than two states. As we later focused on the multi-line cases, we looked first at cases with only two states to get an idea of the functioning of the model, before dropping this restriction and looking at 'any' partially ordered monotonic one-dimensional system.

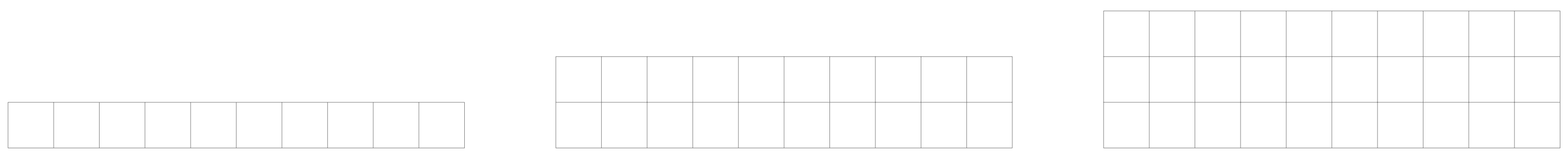

Figure 2: one, two and three line systems with lines in $Z/10 * Z$

Contributions

We got interested in the possibility to find upper and lower bounds on the maximum lifetime of an island in an eroder in any two dimensional systems. A lower bound does exist and we have studied constructive examples that reach this bound, in order to see whether or not it is possible to find a new lower bound or an upper bound, albeit unsuccessfully.

We also tried to adapt results existing for one dimensional systems to the case of two dimensional systems. Rather than trying to adapt the results directly, we wanted to study and understand how the change from a one dimensional space to a nearly-two dimensional space affected the system and its properties. As a one dimensional system is a line and a two dimensional system is a plane composed of an infinity of adjacent lines, we studied cases of systems composed of two adjacent lines, three adjacent lines and also any number of adjacent lines.

We succeeded in expanding Gal'perin's result to two line systems. We also found an adaptation of this result to systems consisting of three lines, where we needed to add conditions to the characterization of eroders. We also tried to prove the result in one direction in a one dimensional partially ordered model that can serve as model for a system consisting of any number of lines.

Conclusions

There still are many open problems in this area. We worked on bridging the gap in knowledge between the line and the plane in the most complete way possible. Studies for the cases with two and three adjacent lines showed that we could maybe make some progress in this direction. We still hope that understanding theses systems can lead to a characterization of the eroder for the two dimensional systems, or at least an upper bound on the lifetime of any island.

We have obtained at least a complete characterization of the eroders for the simple models between the one-dimensional case and the two-dimensional case, and worked on the more general cases. Further research could provide a better characterization for general multi line systems and two dimensional systems, and lead to a better characterization of non-ergodic cellular automata.

Bibliography

[Gacs95] Peter Gàcs. A new version of Tooms proof. Technical report, Department of Computer Science, Boston University, TR 95-009, Boston, MA 02215, 1995.

[Toom] Andrei Toom. Cellular automaton with errors: problems for students of probability, 1995

[Var69] Leonid Vaserstein. Markov Processes over Denumerable Products of Spaces, Describing Large Systems of Automata. Problems of Information Trans- mission, v. 5 (1969), N. 3, pp. 47-52.

[Toom80] Andrei Toom. Stable and Attractive Trajectories in Multicomponent Systems. Multicomponent Random Systems, (R.L.Dobrushin, Ya.G.Sinai, Eds.) Advances in Probability and Related Topics, V. 6 (1980), Dekker, pp. 549-575.

[Toom82] Andrei Toom. [Estimations for measures that describe the behavior of random systems with local interaction. Interactive Markov Processes and their Application to the Mathematical Modelling of Biological Systems]. Pushchino, 1982, pp. 21-33 (in Russian)

[Toom86] Andrei Toom. On Reliable Simulation in Real Time. Preprints of the 1-st World Congress of the Bernoulli Society for Mathematical Statistics and Probability Theory, 1986, v. 2, p. 676.

[Gal76] Gal'perin. One-Dimensional Automaton Networks with Monotonic Local Interaction. Problems Inform. Transmission, 1976, v. 12:4, p. 299–310

[Pet79] N. V. Petri. The unsolubility of the problem of discerning of annuling iterative nets. Researches in the Theory of Algorithms and Mathematical Logic. Moscow, Nauka, 1979 (in Russian).

Acknowledgments

I would like to thank Péter Gacs my tutor for this internship, along with the Boston University and its staff, with whom it was a pleasure to work.